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Coupled Map Lattices as Cellular Automata

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We approach the problem of the complex dynamics of coupled map lattices (CML) by proposing a reduction to deterministic cellular automata (CA) with more than two states per site. The reduction scheme replaces the local map by an approximation in terms of a step function based on a straightforward analysis of the local dynamics. The variation of the spatial coupling in the CML then translates itself as a path in the spaces of rules for the equivalent deterministic CA. The transition to turbulence via spatiotemporal intermittency in the CML is then interpreted as a transition in the space of rules. The observed nonuniversality of this transition can be traced back to the nature of the rules involved on both sides of the transition region and to the character of the escape process from the turbulent state, either strongly deterministic or quasiprobabilistic. The relation between CML, deterministic, and probabilistic CA and the possibility of a mean-field treatment of the dynamics of CML are discussed at a more formal level.

KEY WORDS: Coupled map lattices; cellular automata; spatiotemporal intermittency.

1. INTRODUCTION

In the current literature on the dynamics of extended systems, studies on coupled map lattices (CMLs) occupy an important place (see, e.g., refs. 1). These systems consist of iterative maps (usually of one real variable) coupled together on a local neighborhood $\mathscr{V}_i$ at nodes i of a regular lattice:

$$X_i^{n+1} = f\left(\sum_{j \in \mathscr{V}_i} W_j X_j^n\right) \tag{1}$$

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where the superscripts denote the time and the subscripts the spatial position of the sites, the W_j factors being the coupling weights and f the local map.[2]

Their interest mainly stems from the need for extensive exploratory numerical investigations of systems with many (spatial) degrees of freedom, for which an efficient formalism is still lacking. Indeed, with both space and time discrete but a local phase space which remains continuous, they offer valuable intermediate models between fully continuous nonlinear partial differential equations describing most physical situations and cellular automata (CAs) for which space, time, and the local phase space are all discrete. In particular, an obstacle to the modeling of concrete physical problems in terms of CAs, the "simplest" conceivable complex systems, lies in the absence of continuous control parameters, hindering, for example, the study of important phenomena such as the transition to disorder, which can be overcome in CMLs by varying the weight factors or the shape of the local map.

Qualitative studies[(2,3)] have shown that CMLs may exhibit spatiotemporal patterns very much reminiscent of those which are the landmark of CAs,[(4)] even though they seem intrinsically different from the fully discrete models. In this respect, the representation of *spatiotemporal intermittency* characterized by the sustained coexistence, in space and time, of slowly evolving regular and disordered patches (for an introduction see ref. 5; see also ref. 2) is particularly impressive. This specific turbulent regime has been shown to occur not only in CMLs, but also in partial differential equations[(6)] and in laboratory experiments.[(7)] In a seminal conjecture, Pomeau[(8)] has suggested to set this specific kind of transition from a completely regular state to spatiotemporal intermittency within the framework of the statistical mechanical description of critical phenomena, and to understand the contaminative spreading of the turbulent state as equivalent to a directed percolation process (see ref. 9 for a review article on directed percolation).

Studies developed further to check this conjecture have shown that concepts from the theory of phase transitions and critical phenomena were indeed relevant, but that the exact nature of the transition (continuous or discontinuous) could change with the detailed structure of the CML.[(10)] Moreover, in the case of continuous transitions, critical exponents were seen to depend on the shape of the local map,[(11)] so that CMLs could not belong *stricto sensu* to the universality class of directed percolation as initially conjectured. It was therefore tempting to attack the problem posed by this lack of universality by considering coupled map lattices as cellular

[2] This formulation is equivalent to the usual one: $Y_i^{n+1} = \sum_{j \in \mathscr{V}_i} W_j f(Y_j^n)$.

automata, shifting the debate to the (hopefully) clearer field of fully discrete systems, and trying to understand the apparently strong similarities between these two types of systems at a more formal level. Here, we present first a simple and systematic scheme to construct a deterministic cellular automaton (DCA) approximation of a given CML, using the minimal model introduced previously[11] as an example. The properties of the obtained DCAs are then discussed from the point of view of the reliability of the approximation and, at a more general level, of the formal relation between CMLs and deterministic and probabilistic CAs. The phase transitions of these systems, the existence of universality classes, and the possibility of a mean-field theory are also considered.

2. DETERMINISTIC CELLULAR AUTOMATA APPROXIMATION

The minimal map f chosen to analyze spatiotemporal intermittency was constructed to meet the need of both chaotic transients and a simple linearly stable asymptotic state (e.g., a stable fixed point) possibly unstable to the finite-amplitude perturbations introduced by the coupling. It is piecewise linear and composed of a chaotic repellor (a tent map) connected to a simple attractor (Fig. 1a). It reads

$$\begin{aligned} f(X) &= rX && \text{if } X \in [0, 1/2] \\ f(X) &= r(1-X) && \text{if } X \in [1/2, 1] \\ f(X) &= k(X - X^*) + X^* && \text{if } X > 1 \end{aligned} \tag{2}$$

with $X^* = (r+2)/4$, $|k| \leqslant 1$, and $r > 2$.

In one space dimension, with nearest neighbor diffusive coupling, the CML itself can be specified by introducing the weights $W_i = 1 - \varepsilon$ and $W_{(j=i\pm 1)} = \varepsilon/2$, where ε is the coupling strength. This ensures that the laminar state ($X > 1$) is an *absorbing* state, i.e., that disorder cannot emerge spontaneously from a cluster of "laminar" sites and that the spreading of the turbulent state ($X < 1$) is a contaminative process. Starting from disordered initial conditions, in the infinite-size limit, there exists a precise threshold value ε_c of the coupling below which the system falls into the homogeneous laminar phase and above which sustained regimes of spatiotemporal intermittency are observed and characterized by stationary statistical properties.

The key remark for constructing deterministic cellular automata equivalent to continuous systems is to define CMLs with step functions as local maps (this was incidentally used by Oono and Kohmoto[12]). Therefore, we consider the following reduction of the system defined by (1) with

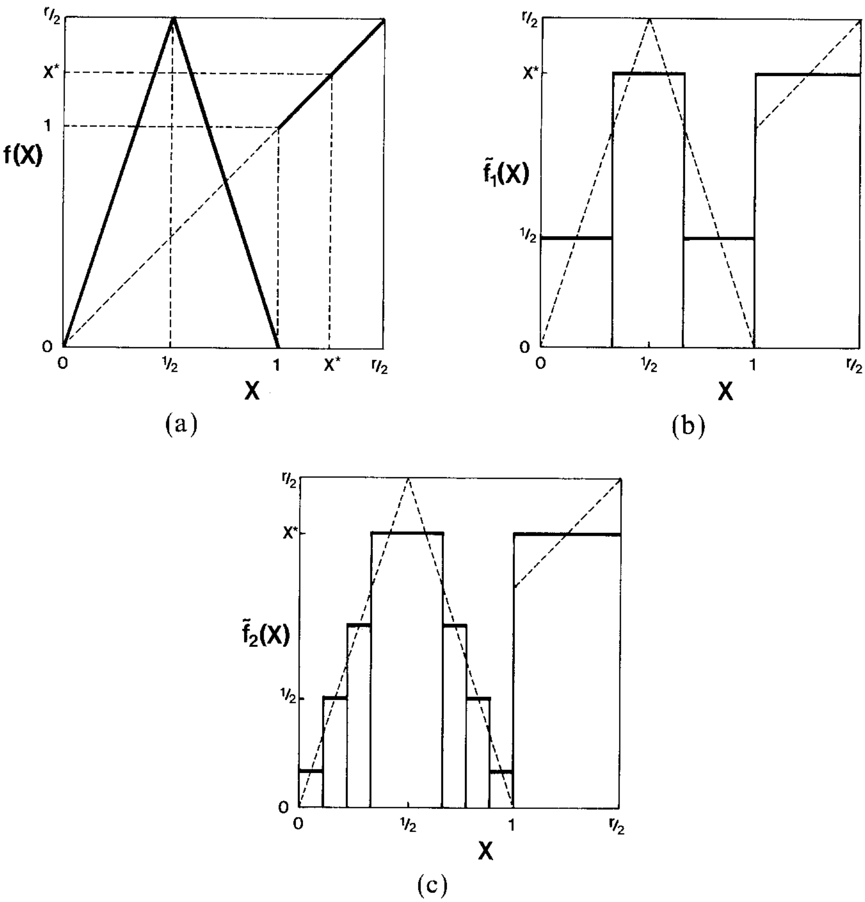


Fig. 1. (a) The elementary map f of which the minimal CML is built. On this figure, $r = 3$, $k = 1$. Also shown are functions derived from the elementary map f with which the DCAs approximating the minimal CML are built (b) at order $p = 1$ (function $\tilde{f}_1$) and (c) at order $p = 2$ (function $\tilde{f}_2$).

f given by (2) in one space dimension, i.e., with weights quoted above, by choosing an approximation to f in terms of a step function $\tilde{f}$ taking k possible values $s_1, s_2,..., s_k$: $\tilde{f}(Y) \in \{s_1, s_2,..., s_k\}$.

The dynamics of f itself is very simple: if X is in the laminar region ($X > 1$) it remains in it, otherwise it can escape the turbulent region ($X < 1$) through the "hole" in the unit interval delimited by the open interval $]1/r, 1 - 1/r[$. This interval I_1 is one of the preimage under f of the laminar region $I_0 =]1, r/2]$, the other being I_0 itself:

$$f^{-1}(I_0) =]1, r/2] \cup]1/r, 1 - 1/r[= I_0 \cup I_1$$

Let J_1 be the complementary set of I_1 in the unit interval:

$$J_1 = [0, 1/r] \cup [1 - 1/r, 1] = [0, 1] - I_1 = [0, r/2] - (I_0 \cup I_1)$$

The above crude description of the dynamics of f is then equivalent to approximating f by the step function $\tilde{f}_1$ defined by (Fig. 1b)

$$\tilde{f}_1(X) = X^* \qquad \text{if} \quad X \in (I_0 \cup I_1)$$

$$\tilde{f}_1(X) = 1/2 \qquad \text{if} \quad X \in J_1$$

This procedure can be easily refined in the following manner: let I_2 be the preimage under f of I_1 and let J_2 be its complement in J_1:

$$I_2 = f^{-1}(I_1)$$

$$J_2 = J_1 - I_2 = [0, r/2] - (I_0 \cup I_1 \cup I_2)$$

Then, $\tilde{f}_2$ will be the step function defined by taking the values by f of all the middle points of the subintervals delimited by the partition (I_0, I_1, I_2) of the invariant interval $[0, r/2]$ (Fig. 1c). The function $\tilde{f}_2$ defines a "better" approximation of f, which is derived from $\tilde{f}_1$ and based on the dynamics of f in a natural manner. It can be generalized at order p by defining the interval

$$I_p = f^{-1}(I_{p-1})$$

$$J_p = J_{p-1} - I_p = [0, r/2] - \bigcup_{q=0}^{p} I_q$$

and building a step function $\tilde{f}_p$ taking the values under f of all the middle points of the subintervals of $[0, r/2]$ defined this way {iterated to infinity, this procedure constructs a Cantor set $\lim_{p \to \infty} J_p$ of fractal dimension $d_f(r) = \log 2/\log[r/(r-2)]$ on the unit interval, which, for $r = 3$, is the well-known triadic Cantor set[13]}.

At order p, $\tilde{f}^p$ is taking $k_p = 2^p$ values and the minimal CML is then approximated by a k_p-state DCA. For every value of the coupling ε, this three-site DCA is governed by a particular rule among the $k_p^{k_p^3}$ available at order p of the approximation. However, this number is reduced in fact to the number of legal rules[14]:

$$\mathcal{N}(p) = k_p^{k_p^2(k_p+1)/2 - 1}$$

since the left/right symmetry of the coupling is preserved and the $X = X^*$ state is an invariant state of the all the $\tilde{f}_p$ step functions $[\tilde{f}_p(X^*) = X^*]$, and hence an absorbing state of the equivalent DCAs. This is in fact the translation under the approximation of the absorbing character of the laminar state ($X > 1$) in the original minimal CML.

Coming back now to the original problem, namely the transition to spatiotemporal intermittency in this minimal CML, we easily see that when ε is varied from, say, 0 to 1, the equivalent DCA changes its rule for particular threshold values of ε, describing a path in the set of the $\mathcal{N}(p)$ possible rules at order p of the approximation. The next section takes a closer look at this path and the dynamics of the associated DCAs along it in order to evaluate the approximation, beginning with the lowest orders.

3. RESULTS

3.1. Approximation at Order $p = 1$

At the lowest order, $p = 1$, the approximated DCA is a two-state, three-site, legal rule automaton. It is well known that there are only $\mathcal{N}(1) = 32$ such rules. Therefore, the equivalent rule at a given ε is easily determined analytically by considering simply the outcome of each of the five possible local configurations (taking into account the left/right symmetry and the absorbing state). The step function $\tilde{f}_1$ takes two values, $s_1 = X^*$ and $s_2 = 1/2$ (Fig. 1b). Since the first one is an absorbing state of the automaton, it is identified with the usual "0" (dead) state in the conventional representation, while the other is the "1" state.

The complete analytical results for $2 \leqslant r \leqslant 3$ (the r values of main interest) are given in Table I. Among the 32 possible rules, only 7 are "visited" when ε is varied from 0 to 1 (except for $r = 3$, where rule 94 disappears). This sequence is nevertheless reminiscent of the actual dynamics of the original CML: there exists a threshold value $\varepsilon_c = 2 - 4/r$ below which the reconstructed automaton is governed by a "trivial" rule (class 1 or 2) and above which the rules are "complex" (class 3).[14] This mimics, already with such a crude approximation, the existence of a threshold for observing sustained spatiotemporal intermittency in the CML. Indeed, in this

Table I. Equivalent Rules for the Deterministic Cellular Automata Approximating the Minimal Coupled Map Lattice at Order $p = 1$ When the Coupling ϵ is Varied between 0 and 1[a]

ε	0		$2/r - 4/r^2$		$1 - 2/r$		$4/r - 8/r^2$		$2 - 4/r$		$2/r$		$1 - 2/r + 4/r^2$		1
Rule		32		36		4		76		94		90		122	
Dynamics						Trivial (class 1 and 2)							Complex (class 3)		

[a] Results valid for $2 \leqslant r \leqslant 3$ only.

context, the familiar patterns developed by class 3 rules[15] are very similar to those exhibited by the CML above the spatiotemporal intermittency threshold (Fig. 2b).

3.2. Approximation at Higher Orders

Considering the approximation at order $p > 1$, analytical evaluation of the equivalent rules when ε is varied becomes rapidly an immense task. Indeed, since the number of states of the DCA is $k_p = 2^p$ at order p, the number of local configurations one has to look at on a three-site neighborhood quickly diverges with p, as seen from the number $\mathcal{N}(p)$ of legal rules $[\mathcal{N}(2) = 4^{39} \sim 10^{27}!]$.

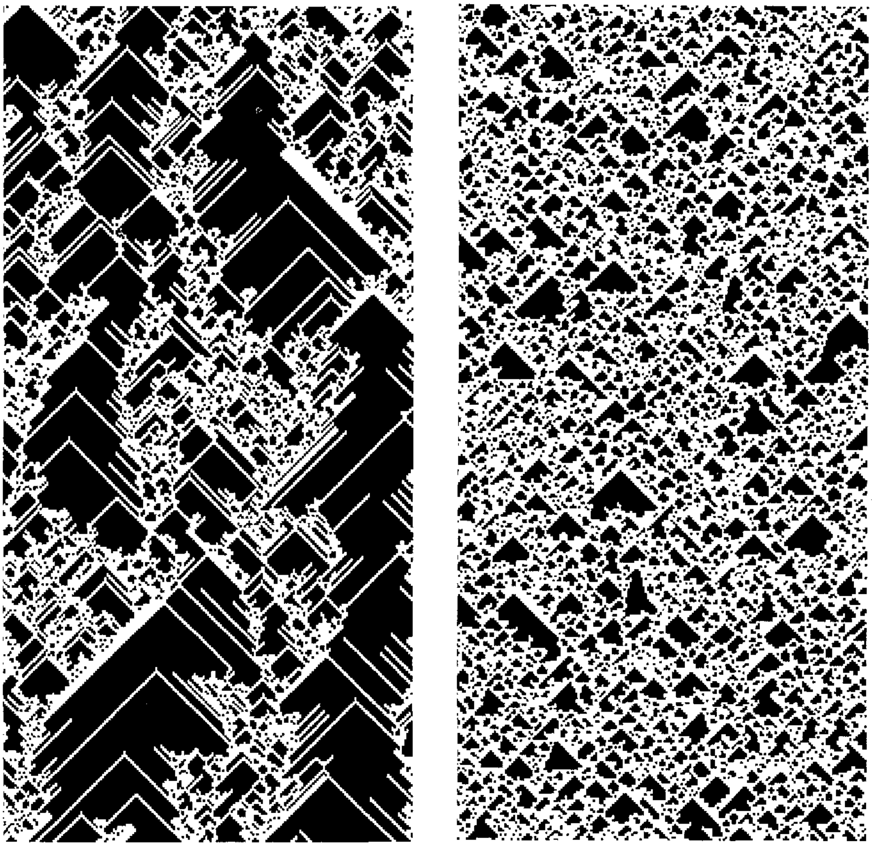
In the same manner, the number of threshold values of ε delimiting the intervals over which the CML is approximated by the same rule increases very fast, roughly like the number of points forming the (uncomplete) Cantor set on which $\tilde{f}_p$ is defined. Therefore, it is not of great interest to specify exactly the equivalent rules, all the more since we are mostly interested in the general qualitative features of the path defined in the set of possible rules by the approximation when ε is varied.

Thus, we turned to numerical simulations of the DCAs approximating the original CML at various orders and mainly for two values of r, $r = 3$ and $r = 2.1$. The transition to spatiotemporal intermittency for the minimal CML has been studied extensively for these values.[11] The corresponding spatiotemporal representations showed very different patterns, the $r = 3$ case being characterized by the propagating structures and triangular clusters of Figs. 2a and 2b, while for $r = 2.1$ the laminar clusters have no well-defined underlying shape, as is the case for directed percolation. These qualitative differences are accompanied by a quantitative discrepancy in the measured critical exponents when the transition is continuous.

In order to get a meaningful spatiotemporal representation of the DCAs at high orders of the approximation, we used the fact that one state among the $k_p = 2^p$ possible states at order p is absorbing, as argued above. As for the CML, a binary reduction of the spatiotemporal information was performed, distinguishing the absorbing state $X = X^*$ from all the other states.

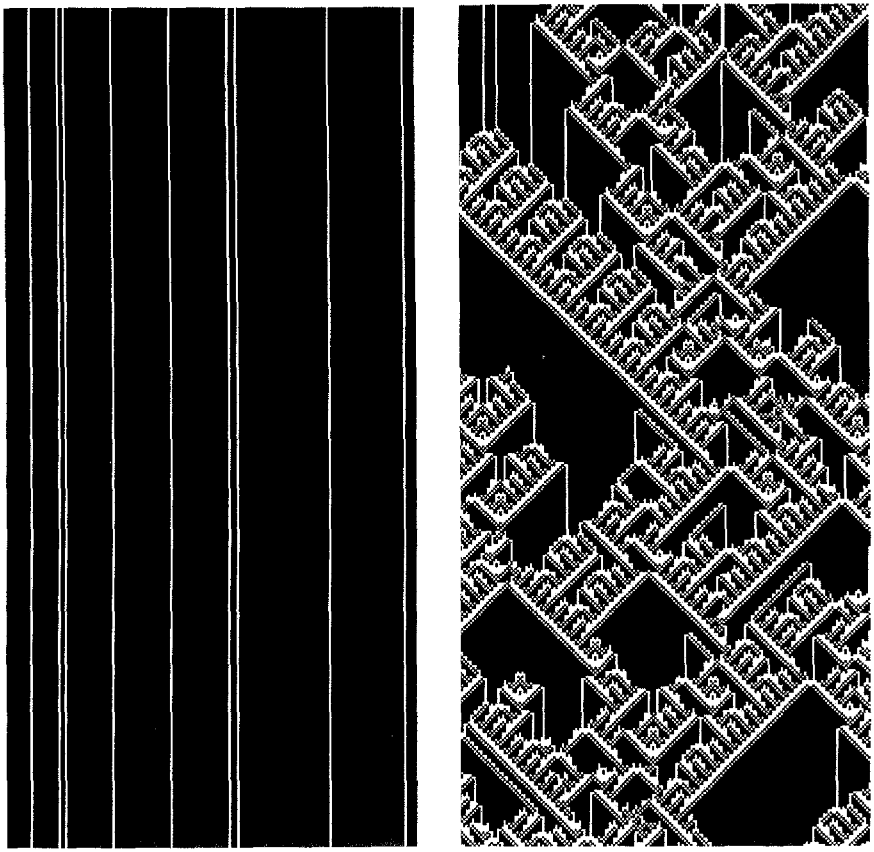
The behavior of the DCA when ε is varied also shows an important difference between the $r = 3$ and the $r = 2.1$ cases for $p > 2$.

For $r = 3$, the general picture drawn for the $p = 1$ case holds at higher orders of approximation. There exists a threshold ε_c of the coupling separating "trivial" rules (class 1 or 2) from complex ones (class 3 or 4). For $\varepsilon < \varepsilon_c$, the behavior of the automaton is simple (frozen spatial structure and stationary or periodic temporal evolution), while it is spatiotemporally intermittent (propagating structures, triangular embeddings) for $\varepsilon > \varepsilon_c$. Figures 2c–2e show the main features of the $p=3$, $r=3$ case. Note that the propagating structures (Fig. 2d), probably corresponding to class 4 rules, appear for ε values around the threshold, as observed for the original CML.[16] Moreover, the threshold ε_c is in closer quantitative agreement with the measured value for the CML, $\varepsilon_c^{\mathrm{CML}} \simeq 0.360$, when p is increased (Table II).



(a) (b)

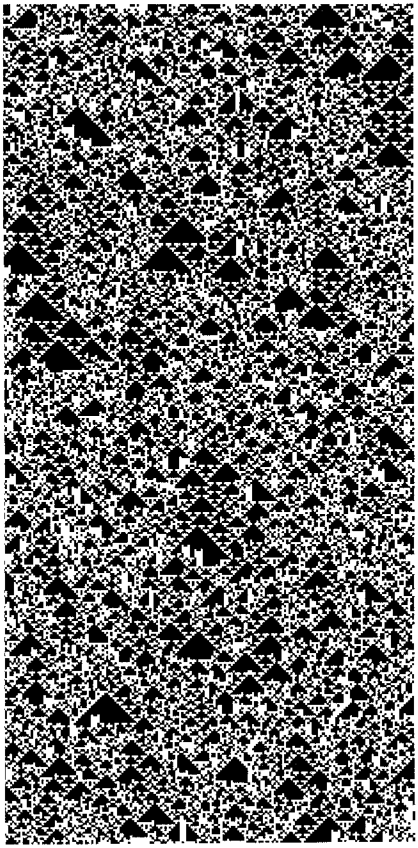
Fig. 2. Spatiotemporal representation of the behavior of the minimal CML for $r=3$ and $k=1$ and its DCA approximation at order $p=3$. Sites in the laminar (absorbing) state are in black, time is running upward, the lattice size is $N=200$ sites, and the boundary conditions are periodic. The evolution is shown during 400 iterations following random initial conditions and a long transient period. (a) CML at the spatiotemporal intermittency threshold, $\varepsilon=0.360$; (b) CML above threshold, $\varepsilon=0.400$; (c) a class 2 DCA rule occurring at $\varepsilon=0.40$; (d) a complex DCA rule with propagating structures (class 4?) occurring at $\varepsilon=0.43$ (threshold region); (e) a class 3 DCA rule occurring above threshold ($\varepsilon=0.55$).



(c) (d)

Fig. 2 (*continued*)

For $r = 2.1$, the general scenario is also respected, although less clearly. The transition is not so well marked, since only class 1 or class 2 rules are observed. Nevertheless, this does not ruin the validity of the approximation. On the contrary, it emphasizes the fact that no particular structure was observed in the spatiotemporal regimes of the CML for this value of r. Being closer to its crisis point, the local map f is better seen as a Poisson-type random generator for the escape time from the turbulent state ($X < 1$). The approximation procedure leads to step functions $\tilde{f}_p$ which remain very close to f itself so that the emerging discrete dynamics keeps a trace of this property. We will come back later to this point when discussing the origin of the nonuniversality of the transition, but we can already emphasize the satisfactory qualitative agreement between the CML and the DCAs derived from the approximation.



(e)

Fig. 2 (*continued*)

Table II. Threshold Values of the Coupling Separating Trivial and Complex Behavior for the DCA Approximating the Minimal CML at $r=3$ at Different Orders p of the Approximation[a]

p	1	2	3	$\geqslant 4$
ε_c	2/3	0.47	0.43	0.36

[a] The threshold for the original CML is $\varepsilon_c^{\mathrm{CML}} \simeq 0.360$.

4. DISCUSSION

4.1. The Transition to Disorder Viewed As a Path in the Space of Rules

The approximation procedure casts the problem of the transition to spatiotemporal intermittency in the minimal CML into a finite set of DCAs whose cardinal $\mathcal{N}(p)$ is rapidly increasing with p, the order of the approximation. When ε is varied, the system no longer describes a (formally) continuous path in the space of CMLs, but a discrete sequence of rules which nevertheless keeps track of the main characteristics of the continuous case. In a certain manner, this sequence defines an order in the *a priori* structureless set of possible rules at given p. "Consecutive" rules may be thought of as "closer" to each other than arbitrarily chosen ones. This stems not only from the fact that the Hamming distance between two such rules is minimal (this distance is simply the number of local configurations whose output is different under the two rules; understandably, a small shift of ε across the threshold value separating the two rules does not change the output of most configurations), but also from a proximity deeply related to the dynamics of the underlying problem.

At lowest order ($p = 1$), the set of possible rules is fairly small [$\mathcal{N}(1) = 32$] and the relative Hamming distance between two consecutive rules rather large (of order 1/32). For higher values of p, this relative distance becomes very small [of order $1/\mathcal{N}(p)$] and the coarseness of the path of rules defined by the approximation progressively vanishes. From this point of view, at a formal level, even the original CML may be considered to be a DCA with a (huge) finite number of states per site when implemented on a digital computer. Indeed, already for p not too large, the length of the smallest intervals defined in [0, 1] by the approximation may be of the order of the precision of the computer on which the system is simulated. We can thus see the path of rules defined at every order, and also by the CML itself, as a line crossing some "critical surface" in the set of possible rules. This critical surface separates rules associated with complex spatiotemporal behavior from those yielding trivial dynamics. At low orders of the approximation, i.e., for DCAs with a small number of possible states per site, there may be eventually no rule lying on or near this surface, so that no "critical behavior" may be observed for a single rule. At higher orders, on the contrary, rules may exist which are sufficiently close to this surface so as to exhibit critical properties, as suggested by the results of the simulations of the CML itself.

4.2. From Deterministic to Probabilistic Cellular Automata

This approach may also be fruitful in understanding *probabilistic* cellular automata (PCAs), for which the output of some local configurations is governed by a random process. Directed percolation is one such system usually derived from purely probabilistic concerns.[9] However, as *every* other PCA, it can be viewed as a probabilized DCA, or, equivalently, as a system defined by a "randomization" between two different DCA rules (see ref. 17 for an example). According to the general picture described above, these two deterministic rules between which a PCA is defined may be thought of as lying on each side of the critical surface in the case where the corresponding PCA indeed shows a phase transition between complex and trivial behavior (usually the case of interest). The randomization process gives relative weights to the two underlying deterministic rules with the help of a (set of) continuous control parameters(s) and the transition threshold corresponds to a critical value of this parameter. We can then think of a PCA as an automaton whose "effective rule" lies somewhere between the two underlying deterministic rules which define it. The critical value of the control parameter would then correspond to this effective rule lying exactly on the critical surface, even though there is possibly no actual deterministic rule on this surface if the number of states per site is too small.

4.3. On the Origin of Nonuniversality

Returning to the approximation described in this work, one can indeed construct PCAs showing critical behavior by probabilizing, at a given order p, between the two deterministic rules lying across the threshold between complex and trivial dynamics (between rules 94 and 90 for $p=1$).[18] In this respect, the $r=3$ and $r=2.1$ cases at higher p discussed above also appear very different from each other:

1. For $r=3$, typical turbulent transients in the local map f remain short; the origin of complex spatiotemporal behavior cannot derive from a local stochastic process, but, on the contrary, from a quasideterministic process translated under the approximation as an interplay between class 1 or 2 rules and complex rules of class 3 or 4, indispensable for sustaining disorder.

2. For $r=2.1$, as argued above, the origin of disorder is rather rooted in the mixing nature of the local dynamics; accordingly, complex spatiotemporal behavior can possibly result from a probabilization between rules of class 1 or 2 only, as in directed percolation, which may be

thought of as being defined between class 1 rules 0 and 254 (on a square lattice).

This important difference is, we believe, at the origin of the nonuniversality observed for the CML, including the discrepancies with directed percolation. It is strengthened by recent results on a specific two-state PCA with one absorbing state, which was shown to possess critical properties at odds with those of directed percolation,[19] contrary to the widespread belief that such automata should all be in the same universality class.[20] Indeed, this system is a PCA defined between a class 3 and a class 1 rule, as the $r = 3$ case in this work. At a more general level, the minimal CML can be seen as an automaton with more than two states per site. These automata are not in the universality class of directed percolation, so that there is no *a priori* reason to find the same critical exponents for both systems.

4.4. Steps for a Mean-Field Approach of CMLs

Finally, the above discussion also suggests a (formal) approach to a mean-field analysis of CMLs. The k_p-state DCA approximating the minimal CML at order p can be reduced to a two-state *probabilistic* automaton by a binary reduction of the type introduced above for the spatiotemporal representations of the dynamics. Such a reduction indeed involves a probabilization of the output of the local configurations, then limited to five (on a three-site neighborhood, with the ordinary restrictions of legality). Mean-field analysis of such automata is straightforward[19,21] and usually approximates the system by an iterative map for the global concentration of active sites (not in the absorbing state). Therefore, the observation of a collective dynamics very similar to that of a single map when performing an "experimental" mean-field treatment (for example, by coupling each site to a large number of "neighbors" chosen at random) on the CML should not be too surprising.[22]

4.5. Conclusion

The proposed procedure to approximate CMLs by DCAs with a small number of possible states per site is very faithful in reproducing most qualitative features of the problem of the transition to spatiotemporal intermittency. It can also be easily extended to higher space dimensions and larger local neighborhoods.

More importantly, it gives hints at explaining the nonuniversality of the transition observed for the minimal CML in relating it to the underlying presence of DCA rules of different classes, an explanation believed to be also valid for recent results on PCAs which also showed nonuniversal behavior[19] (such classes may also have to be subdivided in this respect[3]).

At a more formal level, the procedure suggests redefining the mutual status of CMLs, DCAs, and PCAs, clarifying the respective role of determinism and continuous character as opposed to randomness and discreteness for extended systems.

[3] Results presented in ref. 23 suggest that class 3 rules may have very different properties from the point of view of the various measures of complexity.

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